Tensor decompositions in statistical signal processing

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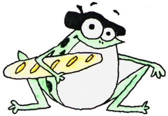
Linear statistical model

$$\mathbf{y} = \mathbf{A}\,\mathbf{s} + \mathbf{b} \tag{1}$$

with

$$\begin{cases} \mathbf{y}: & K \times 1 \text{ random} \\ \mathbf{s}: & P \times 1 \text{ random } \textit{stat. independent} \\ \mathbf{A}: & K \times P \text{ deterministic} \\ \mathbf{b}: & \text{errors} \end{cases}$$



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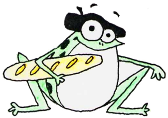

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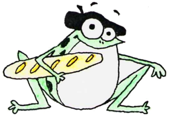

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which is of the *noiseless* form

$$\mathbf{y} = \mathbf{A}\,\mathbf{s} \tag{2}$$

with a *larger* dimension P

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- $\mathbf{A}$ and $\mathbf{s}$ are jointly estimated

Application areas

1. Telecommunications (Cellular, Satellite, Military),
2. Radar, Sonar,
3. Biomedical (EchoGraphy, ElectroEncephaloGraphy, ElectroCardioGraphy)...
4. Speech, Audio,
5. Machine learning,
6. Data mining,
7. Control...

Identifiability & Uniqueness

- Uniqueness/Identifiability up to inherent ambiguities
- Finite number of solutions

Equivalent representations

Let $\mathbf{y}$ admit two representations

$$\mathbf{y} = \mathbf{A}\,\mathbf{s} \quad \text{and} \quad \mathbf{y} = \mathbf{B}\,\mathbf{z}$$

where $\mathbf{s}$ (resp. $\mathbf{z}$) have independent components, and $\mathbf{A}$ (resp. $\mathbf{B}$) have pairwise noncollinear columns.

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- These representations are *equivalent* if every column of $\mathbf{A}$ is proportional to some column of $\mathbf{B}$, and vice versa.
- If all representations of $\mathbf{y}$ are equivalent, they are said to be *essentially unique* (permutation & scale ambiguities only).

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- **Identifiability theorem** $\mathbf{y}$ can be represented as $\mathbf{y} = \mathbf{A}_1\,\mathbf{s}_1 + \mathbf{A}_2\,\mathbf{s}_2$, where $\mathbf{s}_1$ is non Gaussian, $\mathbf{s}_2$ is Gaussian independent of $\mathbf{s}_1$, and $\mathbf{A}_1$ is essentially unique.

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- **Uniqueness theorem** If in addition the columns of $\mathbf{A}_1$ are linearly independent, then the distribution of $\mathbf{s}_1$ is unique up to scale and location indeterminacies.

Example of uniqueness

Let s_i be independent with no Gaussian component, and b_i be independent Gaussian. Then the linear model below is identifiable, but $\mathbf{A}_2$ is not essentially unique whereas $\mathbf{A}_1$ is:

$$\begin{pmatrix} s_1 + s_2 + 2\,b_1 \\ s_1 + 2\,b_2 \end{pmatrix} = \mathbf{A}_1\,\mathbf{s} + \mathbf{A}_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{A}_1\,\mathbf{s} + \mathbf{A}_3 \begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix}$$

with

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{A}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hence the distribution of $\mathbf{s}$ is essentially unique.
But $(\mathbf{A}_1, \mathbf{A}_2)$ not equivalent to $(\mathbf{A}_1, \mathbf{A}_3)$.

Example of non uniqueness

Let s_i be independent with no Gaussian component, and b_i be independent Gaussian. Then the linear model below is *identifiable*, but the distribution of $\mathbf{s}$ is not unique because a 2×4 matrix cannot be full column rank:

$$\begin{pmatrix} s_1 + s_3 + s_4 + 2\,b_1 \\ s_2 + s_3 - s_4 + 2\,b_2 \end{pmatrix} = \mathbf{A} \begin{pmatrix} s_1 \\ s_2 \\ s_3 + b_1 + b_2 \\ s_4 + b_1 - b_2 \end{pmatrix} = \mathbf{A} \begin{pmatrix} s_1 + 2\,b_1 \\ s_2 + 2\,b_2 \\ s_3 \\ s_4 \end{pmatrix}$$

with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Characteristic functions

First c.f.

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Second c.f.

- $\Psi(\mathbf{t}) \stackrel{\text{def}}{=} \log \Phi(\mathbf{t})$
- Properties:
  - Always exists in the neighborhood of 0
  - Uniquely defined as long as $\Phi(\mathbf{t}) \neq 0$

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$$\begin{aligned}\Psi_{x,y}(u,v) &= \log[\mathbb{E}\{\exp \imath(ux+vy)\}] \\ &= \log[\mathbb{E}\{\exp \imath(ux)\}\,\mathbb{E}\{\exp \imath(vy)\}].\end{aligned}$$

□

Problem posed in terms of Characteristic Functions

- If s_p independent and $\mathbf{y} = \mathbf{A}\,\mathbf{s}$, we have the *core equation*:

$$\Psi_y(\mathbf{u}) = \sum_p \psi_p\left(\sum_q u_q\, A_{qp}\right) \tag{4}$$

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- Since s_p independent, $\phi_y(\mathbf{u}) = \prod_p \mathbb{E}\{\exp \imath(\sum_q u_q\, A_{qp}\, s_p)\}$

Function decomposition

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Weierstrass (1885), Stone (1948), Hilbert (1900), Kolmogorov (1957), Sprecher (1965), Hornik (1989), Cybenko (1989)...

- But here, Ψ_y and ψ_p's are characteristic functions.

Problem seen as non symmetric tensor decomposition

Back to core equation (4):

$$\Psi_y(\mathbf{u}) = \sum_p \psi_p \left(\sum_q u_q \, A_{qp} \right), \; \mathbf{u} \in \mathcal{G}$$

- Assumption: functions ψ_p, $1 \leq p \leq P$ admit finite derivatives up to order d in a neighborhood of the origin, containing $\mathcal{G}$.

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- Taking $d = 3$ as a working example:

$$\frac{\partial^3 \Psi_y}{\partial u_i \partial u_j \partial u_k}(\mathbf{u}) = \sum_{p=1}^{P} A_{ip} \, A_{jp} \, A_{kp} \;\; \psi_p^{(3)}(\sum_{q=1}^{K} u_q \, A_{qp})$$

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- of the form $T_{ijk\ell} = \sum_p A_{ip} \, A_{jp} \, A_{kp} \, F_{\ell p}$, if L points $\mathbf{u}_\ell \in \mathcal{G}$.

Problem seen as symmetric tensor decomposition (1/2)

If only the origin in $\mathcal{G}$, *i.e.* $\mathbf{u}_\ell = \mathbf{0}$, then

$$C_{ijk} = \sum_p A_{ip}\, A_{jp}\, A_{kp}\, f_p$$

Multi-linear relation relating *cumulants*.

Problem seen as symmetric tensor decomposition (2/2)

- **Equivalent writing** (still with $d = 3$ as working example)

$$\mathbf{C} = \sum_{p=1}^{P} f_p \, \mathbf{h}(p) \otimes \mathbf{h}(p) \otimes \mathbf{h}(p)$$

where $\mathbf{h}(p) \overset{\text{def}}{=}$ pth column of $\mathbf{A}$.

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Expected rank

For an order-d symmetric tensor of dimension K

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  For what P can one have an exact fit?
- "Expected rank":

$$R(K,d) \stackrel{\text{def}}{=} \left\lceil \frac{\binom{K+d-1}{d}}{K} \right\rceil \tag{5}$$

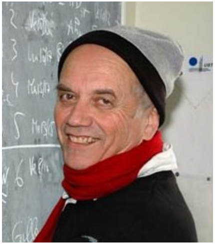
Clebsch's statement

Alfred Clebsch (1833-1872)

For $(d, K) = (4, 3)$, a generic tensor cannot be written as the sum of 5 rank-1 terms, even if $\#unknowns = 15 = \#equations$

Hirschowitz theorem

From Alexander-Hirschowitz thm (1995), one deduces [CGLM08]:



THEOREM For $d > 2$, the generic rank of a dth order symmetric tensor of dimension K is always equal to the expected rank

$$\bar{R}_s = R(K, d) \tag{6}$$

except for $(d, K) \in \{(3, 5), (4, 3), (4, 4), (4, 5)\}$

➡ Only a *finite number* of exceptions !

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- *NS* condition for *almost sure* identifiability:

$$P = \frac{\binom{K+d-1}{d}}{K} = R(K, d) = \bar{R}_s$$

Generic rank of symmetric tensors

Symmetric tensors of order d and dimension K:

d \ K	2	3	4	5	6	7	8
2	2	3	4	5	6	7	8
3	2	4	5	8	10	12	15
4	3	6	10	15	21	30	42

Deterministic approaches

- Model:

$$Y_{ijk} = \sum_p A_{ip}\, s_{jp}\, B_{kp}$$

- $\mathbf{Y} = \sum_{p=1}^{P} \mathbf{a}(p) \otimes \mathbf{s}(p) \otimes \mathbf{b}(p)$

Expected rank

For an order-d symmetric tensor of dimensions K_ℓ, $1 \leq \ell \leq d$

- Number of equations: $\prod_\ell K_\ell$
  Number of unknowns: $\sum_\ell K_\ell P - (d-1)P$

Expected rank

For an order-d symmetric tensor of dimensions K_ℓ, $1 \leq \ell \leq d$

- Number of equations: $\prod_\ell K_\ell$
  Number of unknowns: $\sum_\ell K_\ell P - (d-1)P$
- "Expected rank" again given by the ceil rule:

$$R(K_1, .., K_d) \stackrel{\text{def}}{=} \left\lceil \frac{\prod_\ell K_\ell}{\sum_\ell K_\ell - d + 1} \right\rceil \tag{7}$$

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- For general tensors, generic rank $\bar{R}$ not yet known everywhere.

Generic rank of order 3 free tensors (1)

	K	2				3			4	
	J	2	3	4	5	3	4	5	4	5
I	2	2	3	4	4	3	4	5	4	5
	3	3	3	4	5	5	5	5	6	6
	4	4	4	4	5	5	6	6	7	8
	5	4	5	5	5	5	6	8	8	9
	6	4	6	6	6	6	7	8	8	10
	7	4	6	7	7	7	7	9	9	10
	8	4	6	8	8	8	8	9	10	11
	9	4	6	8	9	9	9	9	10	12
	10	4	6	8	10	9	10	10	10	12
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	12	4	6	8	10	9	12	12	12	13

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- **Thanks for your attention**

Darmois-Skitovich theorem (1953)

Theorem

Let s_i be statistically *independent* random variables, and two linear statistics:

$$y_1 = \sum_i a_i\, s_i \quad \text{and} \quad y_2 = \sum_i b_i\, s_i$$

If y_1 and y_2 are statistically independent, then random variables s_k for which $a_k\, b_k \neq 0$ are Gaussian.

NB: holds in both $\mathbb{R}$ or $\mathbb{C}$

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1 Hence $\sum_{k=1}^{P} \psi_p(u\, a_k + v\, b_k) = \sum_{k=1}^{P} \psi_k(u\, a_k) + \psi_k(v\, b_k)$
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2. Write this at $u + \alpha/a_P$ and $v - \alpha/b_P$:

$$\sum_{k=1}^{P} \psi_k \left(u\, a_k + v\, b_k + \alpha\left(\frac{a_k}{a_P} - \frac{b_k}{b_P}\right)\right) = f(u) + g(v)$$

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3. Subtract to cancel Pth term, divide by α, and let $\alpha \to 0$:

$$\sum_{k=1}^{P-1} (\frac{a_k}{a_P} - \frac{b_k}{b_P})\,\psi_k^{(1)}(u\,a_k + v\,b_k) = f^{(1)}(u) + g^{(1)}(v)$$

for some *univariate functions* $f^{(1)}(u)$ and $g^{(1)}(u)$.

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Conclusion: We have one term less

4 Repeat the procedure $(P-1)$ times and get:

$$\prod_{j=2}^{P}(\frac{a_1}{a_j}-\frac{b_1}{b_j})\,\psi_1^{(P-1)}(u\,a_1+v\,b_1)=f^{(P-1)}(u)+g^{(P-1)}(v)$$

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NB: also holds if ψ_p not differentiable

Definition of Cumulants

- Moments:

$$\mu_r \stackrel{\text{def}}{=} \mathbb{E}\{x^r\} = (-\imath)^r \left. \frac{\partial^r \Phi(t)}{\partial t^r} \right|_{t=0}$$

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- Relationship between Moments and Cumulants obtained by expanding both sides in Taylor series:

$$\log \Phi_x(t) = \Psi_x(t)$$

Polynomial interpolation

Alexander-Hirschowitz Theorem (1995) Let $\mathcal{L}(d, m)$ be the space of hypersurfaces of degree at most d in m variables. This space is of dimension $D(m, d) \stackrel{\text{def}}{=} \binom{m+d}{d} - 1$.

THEOREM Denote $\{p_i\}$ K given distinct points in the complex projective space $\mathbb{P}^m$. The dimension of the linear subspace of hypersurfaces of $\mathcal{L}(d, m)$ having multiplicity at least 2 at every point p_i is:

$$D(m, d) - K(m + 1)$$

except for the following cases:

- $d = 2$ and $2 \leq K \leq m$
- $d \geq 3$ and $(m, d, K) \in \{(2, 4, 5), (3, 4, 9), (4, 1, 14), (4, 3, 7)\}$

In other words, there are a *finite number* of exceptions.

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